# SPIRAL MOTION OF A CYLINDER IN A POWER-LAW NON-NEWTONIAN FLUID

I. P. Grinchik and A. Kh. Kim

UDC 532.135

A study is made of steady-state flow in a power-law system in the presence of double shear caused by spiral motion of a circular cylinder.

The flow of non-Newtonian systems under pure shear has been studied comparatively well. A number of papers are devoted to an experimental [1] and theoretical [2, 3] investigation of the flow of anomalously viscous systems subject to the Schwedow–Bingham equation for the effect of two pure shears.

Let us examine a flow with double shear of non-Newtonian systems described by the Ostwald de Vallee [4] equation. Gutkin [2] solved the problem for a viscoplastic system in such a formulation.

Let the fluid flow in the gap between coaxial cylinders be caused by the uniform spiral motion of the inner circular cylinder. We consider the cylinder to be sufficiently long, and the medium along the cylinder axis to be infinite. Let $v_0$ denote the translational velocity of the cylinder, and $\omega_0$ the angular velocity. Assigning $\omega_0$ and $v_0$ is equivalent to assigning the rotational moment M and the axial force F. The flow of the medium is caused by the cylinder motion, hence, there is no pressure gradient along the cylinder axis.

The rheological equation of state of the system in tensor form is

$$\Pi_0 = 2kh^{n-1}\dot{\Phi}_0. \tag{1}$$

Let us solve the problem in a cylindrical r; z; $\varphi$ coordinate system. The velocity components $v_\varphi$ and $v_z$ depend only on r, and there is no $v_r$. The z axis is directed along the cylinder axis.

Neglecting inertial and mass forces, the Cauchy equilibrium equations can be written as

$$\frac{1}{r^2}\frac{d}{dr}(r^2 p_{r\varphi}) = 0, \tag{2}$$

$$\frac{1}{r}\frac{d}{dr}(r p_{zr}) = 0. \tag{3}$$

The continuity equation is satisfied identically. In conformity with (1), the stress-tensor components different from zero are:

$$p_{r\varphi} = kh^{n-1} r \frac{d\omega}{dr}, \tag{4}$$

$$p_{zr} = kh^{n-1}\frac{dv_z}{dr}, \tag{5}$$

where

$$h = \sqrt{\left(r\frac{d\omega}{dr}\right)^2 + \left(\frac{dv_z}{dr}\right)^2}. \tag{6}$$

It must be kept in mind that

$$\frac{d\omega}{dr} < 0; \quad \frac{dv_z}{dr} < 0; \quad h > 0.$$

In the absence of slip on the boundary with the solid wall the boundary conditions are

Belorussian Polytechnic Institute, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 5, pp. 810-814, May, 1970. Original article submitted July 2, 1969.

$$\text{for}\quad r = r_1 \quad v_\varphi = r_1\omega_0, \quad v_z = v_0;$$
$$\text{for}\quad r = r_2 \quad v_\varphi = 0, \quad v_z = 0.$$

Integrating (2) and (3) we obtain

$$p_{r\varphi} = \frac{C_1}{r^2}, \tag{7}$$

$$p_{zr} = \frac{C_2}{r}, \tag{8}$$

where $C_1$ and $C_2$ are constants of integration. We obtain from (7), (8) and (4), (5) by the Gutkin method [2]

$$kh^{n-1} r \frac{d\omega}{dr} = \frac{C_1}{r^2}, \tag{9}$$

$$\frac{dv_z}{dr} = \alpha r^2 \frac{d\omega}{dr}, \tag{10}$$

where

$$\alpha = \frac{C_2}{C_1}.$$

Formula (6) can be written as

$$h = r\left|\frac{d\omega}{dr}\right|\sqrt{1+\alpha^2 r^2}. \tag{11}$$

Substituting (11) into (9) we obtain

$$\frac{d\omega}{dr} = -\left(\frac{C_1}{k}\right)^{\frac{1}{n}} r^{-\frac{n+2}{n}} (1+\alpha^2 r^2)^{\frac{1-n}{2n}}. \tag{12}$$

Substituting (12) into (10) results in the equation

$$\frac{dv_z}{dr} = \alpha\left(\frac{C_1}{k}\right)^{\frac{1}{n}} r^{\frac{n-2}{n}} (1+\alpha^2 r^2)^{\frac{1-n}{2n}}. \tag{13}$$

The right sides of (12), (13) are binomial differentials. Both equations are integrated in finite form for the values

$$n = \frac{1}{l}, \text{ where } l = 1, 2, 3, \ldots .$$

After separation of variables by the substitution $x = \sqrt{1+\alpha^2 r^2}$, equations (12) and (13) are reduced, respectively, to

$$\omega = -\left(\frac{C_1}{k}\right)^{\frac{1}{n}} \alpha^{2l} \int \frac{x^l dx}{(x^2-1)^{l+1}} + C_3, \tag{14}$$

$$v_z = \left(\frac{C_1}{k}\right)^{\frac{1}{n}} \alpha^{2l-1} \int \frac{x^l dx}{(x^2-1)^{l}} + C_4, \tag{15}$$

where

$$\alpha = \frac{C_2}{C_1} = \frac{F}{M}.$$

The integration constants $C_3$ and $C_4$ are determined from the condition that $\omega = 0$ and $v_z = 0$ for $r = r_2$. For the case of cylinder motion in an infinite fluid, the flow zone is theoretically extended an infinite distance from the cylinder axis, hence, $C_3$ and $C_4$ are determined from the condition that $\omega = 0$ and $v_z = 0$ for $r = \infty$. Recursion formulas can be obtained to evaluate the integrals in (14) and (15). Let us introduce the notation:

$$I_l = \int \frac{x^l dx}{(x^2-1)^l} \quad\text{and}\quad J_l = \int \frac{x^l dx}{(x^2-1)^{l+1}}.$$

Integrating $I_l$ and $J_l$ by parts, we obtain

$$J_l = \frac{x^{l+1}}{(1+l)(x^2-1)^{l+1}} + 2I_{l+2}, \tag{16}$$

$$I_l = \frac{x^{l+1}}{(1+l)(x^2-1)^l} + \frac{2l}{1+l} I_l + \frac{2l}{1+l} J_l. \tag{17}$$

From (16) we obtain

$$I_{l+2} = \frac{-x^{l+1}}{2(1+l)(x^2-1)^{l+1}} + \frac{1}{2} J_l. \tag{18}$$

Analogously from (17) we have

$$J_l = -\frac{x^{l+1}}{2l(x^2-1)^l} - \frac{l-1}{2l} I_l. \tag{19}$$

The integrals can be evaluated directly for $l = 1, 2$.

Finally, we have for $l = 1, 2, 3, \ldots$

$$\omega_l(r) = -\left(\frac{C_1}{k}\right)^l \alpha^{2l} J_l + C_3, \tag{20}$$

$$v_{zl}(r) = \left(\frac{C_1}{k}\right)^l \alpha^{2l-1} I_l + C_4. \tag{21}$$

For other values of $n \neq 1/l$, we evaluate the integrals in (12), (13) approximately. Let us represent the integrands as power series which converge for $|\alpha r| < 1$

$$r^{-\frac{n+2}{n}}(1+\alpha^2 r^2)^{\frac{1-n}{2n}} = r^{-\frac{n+2}{n}} + \frac{1-n}{2n}\alpha^2 r^{\frac{n-2}{n}}$$
$$+ \frac{(1-n)(1-3n)}{2!\,(2n)^2}\alpha^4 r^{\frac{3n-2}{n}} + \frac{(1-n)(1-3n)(1-5n)}{3!\,(2n)^3}\alpha^6 r^{\frac{5n-2}{n}} + \ldots$$

An analogous series is obtained for the integrand of (13), only the exponents of the variable will be two units greater.

The series obtained can be integrated term by term for values $r < M/F$, i.e., when the gap between the coaxial cylinders is not very great and the radius of the inner cylinder is small. After substituting the series obtained into (12), (13) and integrating, we obtain formulas for the angular and translational velocities of the fluid

$$\omega = -\left(\frac{M}{2\pi k}\right)^{\frac{1}{n}} f_1(r, n, \alpha) + C_3',$$

$$v_z = \left(\frac{M}{2\pi k}\right)^{\frac{1}{n}} f_2(r, n, \alpha) + C_4'$$

The integration constants $C_3'$ and $C_4'$ are determined from the boundary conditions.

Let us examine particular cases.

1. If we set $\alpha = 0$ in (12), (13), which corresponds to no axial force F, we then obtain from (12)

$$\frac{d\omega}{dr} = -\left(\frac{C_1}{k}\right)^{\frac{1}{n}} r^{-\frac{n+2}{n}}. \tag{22}$$

It follows from (13) that $v_z = 0$. Since the shear gradient is

$$\dot{\gamma} = r\frac{d\omega}{dr} < 0 \text{ and } \dot{\gamma} = \left(\frac{M}{2\pi k r^2}\right)^{\frac{1}{n}},$$

then we obtain $C_1 = M/2\pi$ from (22), where M is a given torque.

Integrating (22) and determining the integration constant from the boundary conditions, we obtain the fundamental equation (II.2.12) [5] for a coaxially cylindrical viscosimeter

$$\omega_0 = \frac{n}{2}\left(\frac{M}{2\pi k}\right)^{\frac{1}{n}}\left[\left(\frac{1}{r_1}\right)^{\frac{2}{n}} - \left(\frac{1}{r_2}\right)^{\frac{2}{n}}\right]. \tag{23}$$

2. For $r_2 = \infty$ we obtain the torque from (23) which must be applied to a cylinder so that it would rotate uniformly in an infinite fluid

$$M \doteq 2\pi k r_i^2 \left(\frac{2\omega_0}{n}\right)^n.$$

If we put $n = 1$; $k = \mu$, then we obtain the known relations for a Newtonian fluid.

## NOTATION

- $\Pi_0$ is the stress tensor deviator;
- $\Phi_0$ is the strain rate tensor deviator;
- M is the torque;
- F is the axial force;
- $\omega$ is the angular velocity;
- $v_z$ is the translational velocity;
- n is the index of non-Newtonian fluid behavior;
- k is the measure of system consistency;
- $\gamma$ is the shear velocity;
- $v_\varphi$ is the rotational velocity;
- $\omega_0$ is the angular velocity of cylinder;
- $r_1$ is the radius of inner cylinder;
- $r_2$ is the radius of outer cylinder;
- $v_0$ is the translational velocity of cylinder;
- h is the strain rate intensity.

## LITERATURE CITED


1. G. V. Vinogradov, et al., Dokl. Akad. Nauk SSSR, 127, No. 2 (1959).
2. A. M. Gutkin, Kolloidnyi Zhurnal, 22, No. 5 (1960).
3. A. M. Gutkin, Kolloidnyi Zhurnal, 23, No. 1 (1961).
4. M. P. Volarovich and N. I. Malinin, Inzhen.-Fizich. Zh., 16, No. 2 (1969).
5. W. L. Wilkinson, Non-Newtonian Fluid [Russian translation], Mir (1964).